SORPTION IN A PLANE-RADIAL FILTRATION FLOW

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The process of filtration of particles of an admixture is always accompanied by sorption, which in a number of cases, exerts an important influence on the space-time distribution of particles in the flow. The introduction of a number of new processes in the field of petroleum production (such as the "enriching" of the water pumped into a stratum with surface-active substances) requires a quantitative evaluation of the sorption effect. Below we present two forms of exact solution of the sorption problem in a plane-radial flow.

We note that a number of papers [1-7] present results obtained from a study of sorption in one-dimensional filtration flows. The problem considered differs from these investigations not only in relation to the geometry of the flow, but also by taking into account the dependence of the sorption kinetics on filtration rate.

1. Neglecting longitudinal and "convective" diffusion of particles, we can write the continuity equation for a radial flow of sorbate in the form

$$m\frac{\partial c}{\partial t} + \frac{Q}{2\pi r}\frac{\partial c}{\partial r} + \frac{\partial a}{\partial t} = 0. \qquad (1.1)$$

Here $c(r, t)$ is the concentration of sorbate in the liquid, $a(r, t)$ is the adsorption, Q is the volume flow of liquid through an arbitrary cylinder of radius r, m is the porosity of the sorbent.

As usual, we take the kinetic equation in the form [1-3]

$$\partial a/\partial t = \beta(c - c^*). \qquad (1.2)$$

Here β is the kinetic coefficient, c^* is the equilibrium concentration with respect to the adsorption a.

If the concentration of sorbate is low, it is natural to assume that the relationship between a and c^* is linear, that is, corresponds to the Henry isotherm

$$a = Hc^* \text{ (H is the Henry constant).} \qquad (1.3)$$

According to experiments made by L. I. Rubinshtein [5] with calcium and sodium solutions of DS (Soviet detergent) and experiments made at the Ufa Petroleum Scientific Research Institute by L. N. Malysheva with aqueous solutions of the surface-active substance OP-10, the kinetic coefficient β is linearly dependent on filtration rate v

$$\beta = \lambda v \qquad (1.4)$$

or, since in our case $v = Q/2\pi r$

$$\beta = \lambda Q/2\pi r. \qquad (1.5)$$

Assume that a sorbate with concentration c_0 begins to be pumped through a well of radius r at time $t = 0$, and further that at the initial time there is no sorbate either in the liquid phase or in the adsorbed state. Then to equations (1.1)-(1.5) it is necessary to add the conditions

$$c(r, 0) = 0, \qquad a(r, 0) = 0, \qquad c(r_0, t) = c_0. \qquad (1.6)$$

2. In equations (1.1)-(1.5) we made the change of variables

$$\tau = t - \frac{m\pi(r^2 - r_0^2)}{Q}, \qquad \xi = r, \qquad (2.1)$$

which corresponds to a time reading at each point of space from the moment of approach of the leading edge of the sorption wave to this point. Obviously, $c(\xi, \tau)$ and $a(\xi, \tau)$ differ from zero only when $\tau \geq 0$.

Now system (1.1)-(1.5) assumes the form

$$\frac{Q}{2\pi\xi}\frac{\partial c}{\partial \xi} + \frac{\partial a}{\partial \tau} = 0, \qquad \frac{\partial a}{\partial \tau} = \frac{\lambda Q}{2\pi\xi}(c - H^{-1}a). \qquad (2.2)$$

By eliminating the function $a(\xi, \tau)$ we obtain for the dimensionless concentration $u(\xi, \tau) = c(\xi, \tau)/c_0$ the equation

$$\frac{\partial^2 u}{\partial\xi\,\partial\tau} + \lambda\frac{\partial u}{\partial\tau} + \frac{1}{\nu\xi}\frac{\partial u}{\partial\xi} = 0 \qquad \nu = \frac{2\pi H}{\lambda Q} \qquad \begin{pmatrix} 0 < \tau < \infty \\ \xi_0 < \xi < \infty \end{pmatrix}. \qquad (2.3)$$

To this must be added the conditions following from (1.6) and (2.2)

$$u(\xi_0, \tau) = 1, \qquad u(\xi, 0) = e^{-\lambda(\xi - \xi_0)}. \qquad (2.4)$$

It is easy to see that since conditions (2.4) were given on the characteristics of equation (2.3)—the straight lines ξ = const, τ = const, the formulated problem is a Goursat problem for equation (2.3). As is well know, consistency of the initial and boundary conditions at the point ($\xi = \xi_0, \tau = 0$) ensures the existence and uniqueness of the solution of problem (2.3), (2.4).

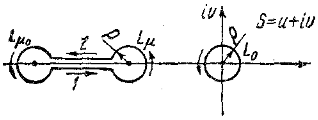

Fig. 1

3. To solve the problem we use the Laplace transformation

$$U(\xi, s) = \int_0^\infty e^{-s\tau} u(\xi, \tau)\, d\tau. \qquad (3.1)$$

Applying (3.1) to equation (2.3) and conditions (2.4), after integration of first-order ordinary differential equation for $U(\xi, s) = u(\xi, \tau)$, we obtain

$$U(\xi, s) = F(\xi, s)\, e^{-\lambda(\xi - \xi_0)}$$
$$F(\xi, s) = \frac{1}{s}\left[\frac{1 + \nu\xi s}{1 + \nu\xi_0 s}\right]^{\lambda/\nu s}. \qquad (3.2)$$

Using the Riemann-Mellin inversion formula, we write

$$u(\xi, \tau) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} e^{s\tau} U(\xi, s)\, ds. \qquad (3.3)$$

Since the function $u(\xi, \tau) \leq 1$, its growth index $\alpha' = 0$; accordingly, an arbitrary positive number can be selected as the integration abscissa α. We introduce the representation

$$F(\xi, s) = \frac{1}{s}\exp\left\{\frac{\lambda}{\nu s}\left[\ln\frac{\xi}{\xi_0} + \ln(s + \mu) - \ln(s + \mu_0)\right]\right\}$$
$$\left(\mu = \frac{1}{\nu\xi}, \quad \mu_0 = \frac{1}{\nu\xi_0}, \quad \mu \leqslant \mu_0\right). \qquad (3.4)$$

It is easy to see that the singular points of $U(\xi, s)$ will be:

(1) $s = 0$ simple pole

$$\left.\begin{array}{l}(2)\ s = -\mu \\ (3)\ s = -r_0\end{array}\right\}\text{logarithmic branch points.} \qquad (3.5)$$

Thus, the function $F(\xi, s)$ is analytic in the half-plane Re $s < \alpha$ if we exclude the pole $s = 0$ and the segment of the real axis $(-\mu_0, -\mu)$, including its end. Obviously, the uniqueness of $F(\xi, s)$ to the left of the point $s = -\mu_0$ is ensured by the presence of two branch points. Since the Jordan lemma is satisfied for $\tau > 0$, we replace direct integration in (3.3) by the contour $L = L_0 + L_{\mu_0} + L_\mu + 1 + 2$, indicated in Fig. 1.

Obviously,

$$\frac{1}{2\pi i}\int_{L_0} e^{s\tau} F(\xi, s)\, ds = e^{\lambda(\xi-\xi_0)}, \qquad \frac{1}{2\pi i}\int_{L_{\mu_0}} e^{s\tau} F(\xi, s)\, ds \to 0 \quad (3.6)$$

with L_{μ_0} contracting to a point.

Further, on the upper edge of the cut 1 the variable $s = xe^{i\pi}$ $(\mu_0 \geqslant x \geqslant \mu + \rho)$

$$J^{(1)} = \frac{1}{2\pi i}\int_1 e^{s\tau} F(\xi, s)\, ds =$$

$$= -\frac{1}{2\pi i}\int_{\mu+\rho}^{\mu_0} \frac{e^{-x\tau}}{x}\left[\frac{\xi(x-\mu)}{\xi_0(\mu_0 - x)}\right]^{-\lambda/\nu x} \exp\frac{i\pi\lambda}{\nu x}\, dx. \quad (3.7)$$

On the lower edge 2 the variable $s = xe^{-i\pi}$ and

$$J^{(2)} = \frac{1}{2\pi i}\int_2 e^{s\tau} F(\xi, s)\, ds =$$

$$= \frac{1}{2\pi i}\int_{\mu+\rho}^{\mu_0} \frac{e^{-x\tau}}{x}\left[\frac{\xi(x-\mu)}{\xi_0(\mu_0 - x)}\right]^{-\lambda/\nu x} \exp\frac{-i\pi\lambda}{\nu x}\, dx. \quad (3.8)$$

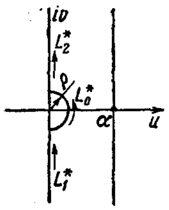

Fig. 2

The sum of the integrals is written

$$J^{(1)+(2)} = -\frac{1}{\pi}\int_{\mu+\rho}^{\mu_0} \frac{e^{-x\tau}}{x}\left[\frac{\xi(x-\mu)}{\xi_0(\mu_0 - x)}\right]^{-\lambda/\nu x} \sin\frac{\pi\lambda}{\nu x}\, dx. \quad (3.9)$$

On the contour L_μ the variable $s = -\mu + \rho e^{i\varphi}$ $(-\pi < \varphi < \pi)$ and

$$J_\mu = \frac{1}{2\pi i}\int_{L_\mu} e^{s\tau} F(\xi, s)\, ds =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\rho e^{\tau s}}{-\mu + \rho e^{i\varphi}}\left[\frac{\xi\rho e^{i\varphi}}{\xi_0(\mu_0 - \mu + \rho e^{i\varphi})}\right]^{\lambda/\nu s} e^{i\varphi}\, d\varphi. \quad (3.10)$$

When $\lambda\xi < 1$ passage to the limit under the integral sign in (3.10) as $\rho \to 0$ is legitimate; this gives

$$\lim J_\mu = 0. \quad (3.11)$$

The integral (3.9) also converges when $\lambda\xi < 1$, and in this case the solution has the form

$$u(\xi, \tau) = 1 - \frac{e^{-\lambda(\xi-\xi_0)}}{\pi}\int_{\mu}^{\mu_0} \frac{e^{-x\tau}}{x}\left[\frac{\xi(x-\mu)}{\xi_0(\mu_0 - x)}\right]^{-\lambda/\nu x} \sin\frac{\pi\lambda}{\nu x}\, dx. \quad (3.12)$$

Assume now that $\lambda\xi > 1$. In (3.10) we pass formally to the limit, retaining only the principal term

$$\lim_{\rho\to 0} J_\mu = \frac{e^{-\mu\tau}\rho^{1-\lambda\xi}}{\pi\mu(\lambda\xi - 1)}\left[\frac{\xi}{\xi_0(\mu_0 - \mu)}\right]^{-\lambda\xi} \sin\pi\lambda\xi. \quad (3.13)$$

After also isolating the principal part of the integral $J^{(1)+(2)}$, which diverges when $\lambda\xi > 1$, we can write

$$\lim J^{(1)+(2)} = -\frac{e^{-\mu\tau}\rho^{1-\lambda\xi}}{\pi\mu(\lambda\xi - 1)}\left[\frac{\xi}{\xi_0(\mu_0 - \mu)}\right]^{-\lambda\xi} \sin\pi\lambda\xi. \quad (3.14)$$

Thus,

$$\lim_{\rho\to 0} J_\mu + \lim_{\rho\to 0} J^{(1)+(2)} = 0. \quad (3.15)$$

Similarly, we find that terms of the order of $\rho^{2-\lambda\xi}$, $\rho^{3-\lambda\xi}$, etc., mutually cancel out. If $\lambda\xi \neq n$ is a whole number, all terms J_μ vanish, which leads to convergence of the expression $J_\mu + J^{(1)+(2)}$. In other words, $(J_\mu + J^{(1)+(2)})$ is a regularization of the integral (3.12), which deverges when $\lambda\xi > 1$. Thus, for arbitrary $\lambda\xi$ the solution can be written in the form

$$u(\xi, \tau) = 1 - \frac{e^{-\lambda(\xi-\xi_0)}}{\pi} R_\mu \int_{\mu}^{\mu_0} \frac{e^{-x\tau}}{x}\left[\frac{\xi(x-\mu)}{\xi_0(\mu_0 - x)}\right]^{-\lambda/\nu x} \sin\frac{\pi\lambda}{\nu x}\, dx. \quad (3.16)$$

Here R_μ is the symbol of regularization of the integral at the lower limit μ. Obviously, when $\lambda\xi < 1$ the regularization is trivial. The singularity of the integrand has the from $(x - \mu)^{-\lambda/\nu x}$ when $x \to \mu$; therefore, we can write

$$z^{-\frac{\lambda\xi}{1+z\mu^{-1}}} = z^{-\lambda\xi(1-z\mu^{-1}+\cdots} = z^{-\lambda\xi}\left(1 + \frac{\lambda\xi}{\mu} z\ln z + \cdots\right)(1 + \cdots) \quad (3.17)$$

that is, the regularized function can be represented in the form of the sum of exponential regularizati ›ns with a decreasing exponent from $\lambda\xi$ to some $\theta < 1$. A method for regularization of functions with exponential singularities was described in [8].

Unfortunately, for quite large $\lambda\xi$ the regularization process leads to extremely unwieldy formulas whose numerical analysis is difficult.

4. In order to obtain $u(\xi, \tau)$ in a form more convenient for computations, we will consider the integration contour L^* shown in Fig. 2. It is easy to confirm that in the region lying between L^* and the straight integration line Re $s = \alpha$ the function $F(\xi, s)$ is analytic, the Jordan lemma conditions are also satisfied, and therefore,

$$u(\xi, \tau) = \frac{1}{2\pi i}\int_{L^*} e^{s\tau} U(\xi, s)\, ds. \quad (4.1)$$

Assuming that at L_1^* $s = -ix$, where $\rho \le x < \infty$, we obtain

$$J_{L_1^*} = \frac{1}{2\pi i}\int_{\infty}^{\rho} e^{-ix\tau}\left[\frac{1 - i\xi\nu x}{1 - i\xi_0\nu x}\right]^{-\lambda/i\nu x} \frac{dx}{x}. \quad (4.2)$$

Similarly, at L_2^* we have $s = ix$ and

$$J_{L_2^*} = \frac{1}{2\pi i}\int_{\rho}^{\infty} e^{ix\tau}\left[\frac{1 + i\xi\nu x}{1 + i\xi_0\nu x}\right]^{\lambda/i\nu x} \frac{dx}{x}. \quad (4.3)$$

The integral for a semicircle of radius ρ is equal to half the residue of the function $F(\xi, s)$ at the point $s = 0$ and therefore has the form

$$J_{L_0^*} = {}^1/_2 e^{\lambda(\xi-\xi_0)}. \quad (4.4)$$

Using the notation

$$1 + i\xi\nu x = R_1 e^{i\varphi_1} \qquad (R_1 = (1 + \xi^2\nu^2x^2)^{1/2} \quad \varphi_1 = \operatorname{arc\,tg} \xi\nu x)$$
$$1 + i\xi_0\nu x = R_2 e^{i\varphi_2} \qquad (R_2 = (1 + \xi_0^2\nu^2x^2)^{1/2} \quad \varphi_2 = \operatorname{arc\,tg} \xi_0\nu x) \qquad (4.5)$$

and adding (4.2), (4.3) and (4.4), after simplification we obtain

$$u(\xi, \tau) = \frac{1}{2} + \frac{e^{-\lambda(\xi-\xi_0)}}{\pi}\int_0^\infty \exp\frac{\lambda(\varphi_1 - \varphi_2)}{\nu x}\sin\left(x\tau - \frac{\lambda R}{\nu x}\right)\frac{dx}{x} \qquad \left(R = \ln\frac{R_1}{R_2}\right). \qquad (4.6)$$

In the case of a point source ($\xi_0 = 0, \varphi_2 = 0, R_2 = 1, \varphi = \varphi_1, R = \ln R_1$) we have

$$u(\xi, \tau) = \frac{1}{2} + \frac{e^{-\lambda\xi}}{\pi}\int_0^\infty \exp\frac{\lambda\varphi}{x}\sin\left(\frac{x\tau}{\nu} - \frac{\lambda R}{x}\right)\frac{dx}{x}. \qquad (4.7)$$

It is easy to check that the convergence of the integrals (4.6) and (4.7) is ensured, since when $x = 0$ and $x = \infty$ the integrand behaves as $x^{-1} \sin kx$.

In numerical computations it is convenient to divide the integration interval into two parts $(0, A)$ and (A, ∞) in such a way that the integral for the second can be discarded or evaluated from integral sine tables. It is convenient to use the Filon formulas [9] for numerical integration in the first interval for large τ/ν.

As an example, Fig. 3 shows the results of computations of $u(\tau)$ using formula (4.7) for the case $\xi = 10$ m, $\lambda = 0.1$ m^{-1}, $\nu = 4$ days/m.

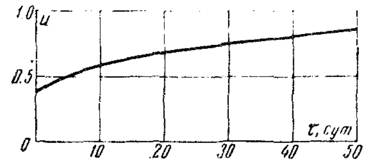

Fig. 3

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